Duality and cohomology in M-theory with boundary

Hisham Sati *

Department of Mathematics
University of Maryland
College Park, MD 20742

Abstract

We consider geometric and analytical aspects of M-theory on a manifold with boundary Y^{11}. The partition function of the C-field requires summing over harmonic forms. When Y^{11} is closed Hodge theory gives a unique harmonic form in each de Rham cohomology class, while in the presence of a boundary the Hodge-Morrey-Friedrichs decomposition should be used. This leads us to study the boundary conditions for the C-field. The dynamics and the presence of the dual to the C-field gives rise to a mixing of boundary conditions with one being Dirichlet and the other being Neumann. We describe the mixing between the corresponding absolute and relative cohomology classes via Poincaré duality angles, which we also illustrate for the M5-brane as a tubular neighborhood. Several global aspects are then considered. We provide a systematic study of the extension of the E_8 bundle and characterize obstructions. Considering Y^{11} as a fiber bundle, we describe how the phase looks like on the base, hence providing dimensional reduction in the boundary case via the adiabatic limit of the eta invariant. The general use of the index theorem leads to a new effect given by a gravitational Chern-Simons term CS_{11} on Y^{11} whose restriction to the boundary would be a generalized WZW model. This suggests that holographic models of M-theory can be viewed as a sector within this index-theoretic approach.

*e-mail: hsati@math.umd.edu

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1 Introduction

M-theory has proven to be a rich theory both in terms of modeling physical phenomena and in terms of mathematical structures. Physical and mathematical insight could be gained by studying various aspects of this theory. In this paper we study geometric and analytical aspects of M-theory on a manifold with a boundary, building on [14], mainly by emphasizing importance of boundary conditions and their effect on the corresponding bundles, fields, actions and partition functions. The main ingredient we use for the kinetic term is harmonic forms. In the presence of a boundary, the Hodge decomposition theorem has to be modified and new effects appear, depending on boundary conditions. For the phase we use index theory on manifolds with a boundary, departing from [14] and [18] by the use of the adiabatic limit of the eta-invariants.

We take M-theory on an 11-dimensional Spin manifold with boundary Y^{11} equipped with a Riemannian metric g_Y. The main fields we consider are the C-field C_3 with its field strength G_4 as well the dual field G_7, the 11-dimensional Hodge dual to G_4 at the level of differential forms. The C-field has a classical harmonic part (see e.g. [28]), which is characterized in [37] in the extension to the Spin bundle. The Bianchi identity and equation of motion for the C-field in M-theory, which follow from those of eleven-dimensional supergravity [11], are

$$dG_4 = 0\ , \qquad \frac{1}{\ell_p^3} d * G_4 = \frac{1}{2} G_4 \wedge G_4 - I_8, \tag{1.1}$$

where I_8 is the one-loop polynomial, $*$ is the Hodge duality operation in eleven dimensions, and ℓ_p is the scale in the theory called the Planck 'constant'. A formulation in terms of G_4 and G_7 is given in [10]. The presence of $d * G_4 = dG_7$ suggests looking also at a degree eight field G_8 (this is called Θ in [14]). The two fields G_4 and G_8 can be treated in a unified way [32] [33] [34] [37].

Harmonic C-field. The classical (or low energy) limit is obtained by taking $\ell_p \to 0$ and is dominated by the metric-dependent terms. In this long distance approximation of M-theory one keeps only the harmonic modes of the C-field [28] [37]. Let $\Delta_g^3 : \big(\Omega^3(Y^{11}), g\big) \longrightarrow \big(\Omega^3(Y^{11}), g\big)$ be the Hodge Laplacian on 3-forms on the base Y^{11} with respect to the metric g given by $\Delta_g^3 = d\, d^* + d^* d$, where d^* is the adjoint operator to the de Rham differential operator d. Assuming $[G_4] = 0$ in $H^4(Y^{11}; \mathbb{R})$ so that $G_4 = dC_3$ then in the Lorentz gauge, $d^* C_3 = 0$, we have [37] $\Delta_g^3 C_3 = *j_e$, where j_e is the electric current associated with the membrane given by $j_e = \ell_p^3 \left(\frac{1}{2} G_4 \wedge G_4 - I_8\right)$. Thus, C_3 is harmonic if $\ell_p \to 0$ and/or there are no membranes. The space of harmonic 3-forms on Y^{11} is $\mathcal{H}_g^3(Y^{11}) := \ker \Delta_g^3 \subset \Omega^3(Y^{11})$. Harmonic forms are very important in compactification, where the fields are expanded in a harmonic basis. For instance, if α^i is basis for the space $\mathcal{H}^3(Y^{11})$ of harmonic 3-forms on Y^{11} then the C-field can be expanded as $C_3 = \sum_i C_3^i \alpha^i$. There are natural choices for internal manifolds for compactifications with fluxes leading to supersymmetric theories in lower dimensions (see [16] and references therein). A seven-dimenisonal manifold M with a 3-form φ is a

G_2 manifold if $d\varphi = d^*\varphi = 0$, that is if φ is harmonic. An eight-manifold with a self-dual four-form $\phi = *\phi$ is called a torsion-free Spin(7) manifold if $d\phi = 0$.

The C-field in the presence of a boundary. When Y^{11} has a boundary we no longer assume that there is a bounding twelve-manifold Z^{12}. The topological sectors of the C-field are labelled by extensions $\tilde{a}$ of the degree four characteristic class of the C-field on $M^{10} = \partial Y^{11}$. In addition to summing over torsion, there will be an integral over a certain space of harmonic fields. Consider the inclusion $i : M^{10} \hookrightarrow Y^{11}$, which induces the pullback on cohomology $i^* : H^4(Y^{11};\mathbb{Z}) \to H^4(M^{10};\mathbb{Z})$. In [14] the sum over the topological sectors in the wavefunction is restricted to $\ker i^* \subset H^4(Y^{11};\mathbb{Z})$, which is equivalent to a sum over $H^4(Y^{11}, M^{10};\mathbb{Z})/\delta H^3(M^{10};\mathbb{Z})$, where δ is the connecting homomorphism, and the integration in the path integral would be over the compact space of harmonic forms $\mathcal{H}^3(Y^{11}, M^{10})/\mathcal{H}^3(Y^{11}, M^{10})_{\mathbb{Z}}$, where $\mathcal{H}^3(Y^{11}, M^{10}) := \ker i^*$ restricted to $\mathcal{H}^3(Y^{11})$. It is desirable to further characterize these, which is one of the goals of this paper. We formulate a boundary value problem which is solvable from general considerations in section 2.2. We work with C_3 as well as its field strength so that both degree three and four cohomology are relevant.

Boundary conditions and duality. In the absence of the field dual to the C-field, the boundary conditions can be taken in a straightforward way. However, when this field is introduced, an interplay between Hodge duality and dynamics of the fields makes such obvious choices not possible. In particular, if the C-field is taken to satisfy the Dirichlet boundary condition then its dual must satisfy the Neumann boundary condition, and vice versa. Thus in this paper we provide a systematic study of these matters. Naturally, then one might ask what replaces the duality in Y^{11} when one restricts to the boundary. We study analogs of the Hilbert transform introduced in [4] which effectively provides a description for such a duality and exchanges Dirichlet and Neumann forms. In addition, we will consider generalization to include the dual fields in section 2.1.

Cohomology in the presence of boundary. An arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form, i.e. the natural map $\mathcal{H}^k(M) \to H^k_{dR}(M)$ is an isomorphism. This means that every cohomology class contains exactly one harmonic form. When Y^{11} is closed then, from the Hodge decomposition theorem, the fourth cohomology group with real coefficients $H^4(Y^{11};\mathbb{R})$ is isomorphic to the space of closed and coclosed differential 4-forms on Y^{11}. Thus, the space of these harmonic forms provides a concrete realization of the cohomology group $H^4(Y^{11};\mathbb{R})$ inside the space $\Omega^4(Y^{11})$ of all 4-forms on Y^{11}. The Laplacian Δ on p-forms on a closed Y^{11} is self-adjoint. However, in the presence of a boundary this is no longer the case, and in fact Δ is surjective [7]. In this case we use (in section 2.2) the Hodge-Morrey-Friedrich (HMF) decomposition theorem [29] [19] which gives us a concrete realization of the absolute cohomology $H^4(Y^{11};\mathbb{R})$ and the relative cohomology $H^4(Y^{11}, \partial Y^{11};\mathbb{R})$ inside the space of all harmonic 4-forms on Y^{11}. The two spaces, surprisingly, intersect only at zero (see [41]) $\mathrm{Harm}^4(Y^{11}, \partial Y^{11};\mathbb{R}) \cap \mathrm{Harm}^4(Y^{11};\mathbb{R}) = \{0\}$. In addition, the boundary subspace of each is orthogonal to all of the other. Each of $H^4(Y^{11};\mathbb{R})$ and $H^4(Y^{11}, \partial Y^{11};\mathbb{R})$ has a portion consisting of those cohomology classes coming from the boundary ∂Y^{11} and another portion of those coming from the interior part of Y^{11}. The principal angles between the interior subspaces of the concrete realizations of $H^4(Y^{11};\mathbb{R})$ and $H^4(Y^{11}, \partial Y^{11};\mathbb{R})$ are called *Poincaré duality angles* and are characterized, via a refinement of the Hodge-Morrey-Friedrich decomposition, in [13] and also [41]. Poincaré duality angles measure how near a manifold with boundary is to being closed. We will be interested in orthogonal decomposition and not just direct sum decomposition (kinetic terms etc.). Similar discussion is provided for other fields, namely C_3, G_7 and G_8. We highlight new effects on the fields due to such phenomena. All of this is discussed in section 2.3. Considering the M5-brane as a tubular neighborhood in the ambient spacetime we illustrate, in section 2.4, the dependence of kinetic terms on distance scales.

E_8 gauge theory for $\partial Y^{11} \neq \emptyset$. The phase of the (non-gravitational) partition function can be studied using E_8 gauge theory [15]. For each characteristic class $a \in H^4(Y^{11};\mathbb{Z})$ of an E_8 bundle over Y^{11} there is a har-

monic four-form G_4^a of the appropriate topological class. The kinetic energy $|G_4^a|^2 = \int_{Y^{11}} G_4^a \wedge * G_4^a$ vanishes if and only if G_4^a is torsion. The partition function involves evaluating the sum $\sum_{a \in H^4(Y^{11};\mathbb{Z})} (-1)^{f(a)} \exp\left(-|G_4^a|^2\right)$, where $f(a)$ is a quadratic refinement of a bilinear form related to a [15]. In dealing with boundaries one has to impose boundary conditions on the C-field within the E_8 model. In [14] the conditions $i^*(C) = 0$ is chosen, where $\check{C} = (A, C)$, A is a connection on an E_8 bundle. This restricts to the C-fields $\mathrm{E}_P(Y^{11}, M^{10}) := \{(A, C) \in \mathrm{E}_P(Y^{11}) \mid i^*(C) = 0\}$. The boundary condition breaks the topological gauge symmetry $\mathcal{G}$, which is breaking of groupoid rather than of structure groups. This implies the following for the kinetic terms of the E_8 gauge field strength F [14]. While the term $\int_{Y^{11}} \mathrm{tr} F \wedge * F$ is not gauge invariant, and hence has no physical degrees of freedom in the interior, the corresponding term $\int_{M^{10}} \mathrm{tr} F \wedge * F$ on the boundary is gauge invariant and hence defines dynamical E_8 gauge fields there. Some supersymmetric aspects of this are discussed in [17]. The generalization of the boundary condition on the C-field leads to conditions for extension of the E_8 bundle, which we also characterize in section 3.1. This generalizes the abelian case in four dimensions [45] to the E_8 case in eleven dimensions (but the discussion holds in more dimensions).

The effective action and partition function. The exponentiated action in the closed case is [15]

$$\exp\left[-2\pi \int_{Y^{11}} \frac{1}{\ell_p^9} \mathrm{vol}(g_Y) R(g_Y) + \frac{1}{2\ell_p^3} G_4 \wedge * G_4\right] \Phi_{RS} \cdot \Phi(C_3) , \tag{1.2}$$

where Φ_{RS} is the Rarita-Schwinger contribution from the Dirac form $\overline{\psi} D \psi$, and $\Phi(C_3)$ is the phase built out of the topological parts of the action, namely the Chern-Simons term and the one-loop term. This phase is studied extensively in [15] for the case of no boundary and in [14] when Y^{11} has a nonempty boundary. The phase also leads to interesting topological structures [39] [37] [38]. In this paper we concentrate both on the phase and on the terms involving ℓ_p, and in particular the kinetic term for the C-field. We can see that when ℓ_p is small, corresponding to a semiclassical approximation, the kinetic term for the C-field will dominate the exponential in (1.2) while the contribution from the Einstein-Hilbert term ($R(g_Y)$ is the scalar curvature) will be very small in comparison. This can also be seen from the measure. The measure for the C-field path integral is [14] $\mu(C_3) \cdot \mathrm{Pfaff}(D_{RS}) \cdot \Phi(C_3) \exp[-\frac{1}{\ell_p^3} \int_{Y^{11}} G_4 \wedge * G_4]$, where $\mu(C_3)$ is the standard formal measure for 3-form gauge potentials defined by the metric g_Y on Y^{11} together with the Faddeev-Popov procedure applied to small C-field gauge transformations. As in [14] we also consider a fixed metric on Y^{11}. In addition to the terms in (1.2) there are terms corresponding to four-fermion interactions as well as interactions of fermions with the C-field. Aspects of the latter is considered in [17] and more fully in [25].

The phase via the adiabatic limit in the boundary case. The boundary conditions for the Dirac operator and the corresponding Pfaffians on Y^{11} with boundary are discussed in detail in [18]. M-theory is shown to be well-defined topologically on an arbitrary number of spatial components. The partition function is constructed in the presence of boundaries using exponentiated eta-invariants. What we do (in section 3.2) is provide an interpretation using the adiabatic limit as in the case of no boundaries [31] [35] [38]. Take Y^{11} to be the total space of a bundle with fiber N^n and base spaceX^{11-n}. Assuming $\partial Y^{11} \neq \emptyset$, there are two cases to consider. First, that the base has a boundary, in which case we make use of the results of Dai [12]. Second, that instead the fiber has a boundary, for which we realize the constructions of Bismut-Cheeger [5] [6] and of Melrose-Piazza [27]. The adiabatic limit, via the point of view advocated in [31] [35] [38], amounts to a dimensional reduction which keeps track of the geometry and analysis involved. Thus, the final expressions will be ones on the base X^{11-n}.

Geometric corrections to the index and the gravitational Chern-Simons term. Local boundary conditions (Dirichlet) can be imposed for the de Rham complex. However, for the Spin and signature complexes, this can no longer be done due to topological obstructions to finding local boundary conditions. Instead, one uses the spectral boundary condition determined by the spectrum of operators on the boundary [1]. Furthermore, in the case of differential forms with duality, as is the case with the C-field and its dual

described by the signature complex [38], we can no longer impose Dirichlet boundary conditions on both the C-field and its dual. So there will be an exclusion principle. Furthermore, the proof in [1] of the index theorem with boundary assumes that the Riemannian manifold has a product metric near the boundary. For general manifolds, there is a correction form given by a gravitational Chern-Simons term. In [21] Horava proposed a holographic nonperturbative description of M-theory via a local quantum field theory. This involves supersymmetric extension of the bosonic Chern-Simons gravity Lagrangian of Chamseddine [8] [9]. In section 3.3, we show that the bosonic part can be obtained by a careful application of the index theorem, following the construction of Gilkey [20]. We obtain this term in addition to what is already present in the index-theoretic description of the action, so this suggests that the proposed holographic and field-theoretic description of M-theory represents a sector within the general description via index theory.

This paper mainly takes [14] as a setting and starting point, applies and physically models the mathematical constructions of [13] [41] [6] [27], and extends and generalizes the geometric constructions in [45].

2 Local aspects: The fields on an eleven-manifold with boundary

We start with a smooth, closed, oriented Riemannian eleven-manifold Y^{11} with a Riemannian metric g_Y. Consider $\Omega^p(Y^{11})$, the space of p-forms on Y^{11}. The operators relevant for us are the de Rham differential $d : \Omega^p(Y^{11}) \to \Omega^{p+1}(Y^{11})$, the Hodge $*$-operator $* : \Omega^p(Y^{11}) \to \Omega^{11-p}(Y^{11})$, the co-differential (the adjoint of d) given by $d^* = (-1)^p * d* : \Omega^p(Y^{11}) \to \Omega^{p-1}(Y^{11})$, and the Hodge Laplacian $\Delta = dd^* + d^* d : \Omega^p(Y^{11}) \to \Omega^p(Y^{11})$. Define the following subspaces of $\Omega^p(Y^{11})$. The spaces of exact p-forms and co-exact p-forms on a closed Y^{11} are, respectively,

$$\begin{aligned} \mathcal{E}^p(Y^{11}) &:= \{\omega \in \Omega^p(Y^{11}) \ : \ \omega = d\eta \text{ for some } \eta \in \Omega^{p-1}(Y^{11})\} , && (2.1)\\ c\mathcal{E}^p(Y^{11}) &:= \{\omega \in \Omega^p(Y^{11}) \ : \ \omega = d^* \xi \text{ for some } \xi \in \Omega^{p+1}(Y^{11})\} . && (2.2) \end{aligned}$$

Furthermore, on a closed Y^{11} there is no distinction between harmonic p-fields

$$\mathrm{Harm}^p(Y^{11}) := \{\omega \in \Omega^p(Y^{11}) \ : \ d\omega = 0 \text{ and } d^*\omega = 0\} \tag{2.3}$$

and harmonic p-forms

$$\mathcal{H}^p(Y^{11}) := \{\omega \in \Omega^p(Y^{11}) \ : \ \Delta\omega = 0\} . \tag{2.4}$$

The L^2 inner product on $\Omega^p(Y^{11})$ is given by $\langle \alpha, \beta \rangle_{L^2} = \int_{Y^{11}} \alpha \wedge *\beta$, for $\alpha, \beta \in \Omega^p(Y^{11})$. When Y^{11} has no boundary, the cohomology of Y^{11} is given, via the classical de Rham theorem, by the cohomology of the de Rham complex $0 \to \Omega^0(Y^{11}) \xrightarrow{d} \Omega^1(Y^{11}) \xrightarrow{d} \Omega^2(Y^{11}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{10}(Y^{11}) \xrightarrow{d} \Omega^{11}(Y^{11}) \xrightarrow{d} 0$, that is there is an isomorphism $H^p(Y^{11};\mathbb{R}) \cong (\ker(d) : \Omega^p(Y^{11} \to \Omega^{p+1})/(\mathrm{im}(d) : \Omega^{p-1}(Y^{11}) \to \Omega^p(Y^{11}))$. The exterior differential commutes with the Laplacian Δ and hence preserves harmonicity of forms, so that one can build a subcomplex $(\mathrm{Harm}^*(Y^{11}), d)$ of the de Rham complex, called the *harmonic complex* in [7],

$$0 \to \mathrm{Harm}^0(Y^{11}) \xrightarrow{d} \mathrm{Harm}^1(Y^{11}) \xrightarrow{d} \mathrm{Harm}^2(Y^{11}) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Harm}^{10}(Y^{11}) \xrightarrow{d} \mathrm{Harm}^{11}(Y^{11}) \xrightarrow{d} 0 . \tag{2.5}$$

Being harmonic is equivalent to being closed and co-closed on an eleven-manifold without a boundary, in which case the maps in the harmonic complex are all zero, and by Hodge's theorem $H^p(\mathrm{Harm}^*(Y^{11}), d) = \mathrm{Harm}^p(Y^{11}) \cong H^p(Y^{11};\mathbb{R})$.

2.1 Hodge theory on the boundary

Taking our eleven-manifold to have boundary, several new phenomena occur. First, the space of harmonic p-fields no longer coincide with the space of harmonic p-forms. Second, the Hodge decomposition theorem is modified as mentioned in the introduction. Third, the space of harmonic p-fields $\mathrm{Harm}(Y^{11})$ is infinite dimensional, and so is too big to represent cohomology. We will illustrate these ideas in the setting of the M-theory fields in what follows.

Relating harmonic potentials to harmonic field strengths on the boundary. Now consider Y^{11} to have a non-empty boundary ∂Y^{11}. In this case the cohomology of the complex $(\mathrm{Harm}^*(Y^{11})), d)$ of harmonic forms on Y^{11} is given by the direct sum $H^p(\mathrm{Harm}^*(Y^{11}), d) \cong H^p(Y^{11};\mathbb{R}) + H^{p-1}(Y^{11};\mathbb{R})$, for $p = 0, 1, \cdots, 11$ [7]. This allows us to characterize the cohomology of Y^{11} in two consecutive degrees in terms of the harmonic cohomology in one degree. We are interested in the pairs of degrees $(3, 4)$ and $(7, 8)$, representing the pairs of fields (C_3, G_4) and (G_7, G_8). For instance, for the first pair we have

$$
\begin{aligned}
H^4(\mathrm{Harm}^*(Y^{11}), d) = \frac{\ker(d) : \mathrm{Harm}^4(Y^{11}) \to \mathrm{Harm}^5(Y^{11})}{\mathrm{im}(d) : \mathrm{Harm}^3(Y^{11}) \to \mathrm{Harm}^4(Y^{11})} \quad &\cong \quad H^4(Y^{11};\mathbb{R}) + H^3(Y^{11};\mathbb{R}) \\
\{G_4 \mid dG_4 = 0, \Delta G_4 = 0, \nexists\, C_3 \text{ with } \Delta C_3 = 0 \text{ so that } G_4 \neq dC_3\} \quad &= \quad \{G_4 \mid dG_4 = 0, G_4 \neq dC_3\} \\
&+ \quad \{C_3 \mid dC_3 = 0, C_3 \neq dB_2\} \ ,
\end{aligned}
$$

where B_2 is 2-form on Y^{11}. Then we have

Proposition 1 *Consider M-theory on a manifold with a boundary. Harmonic closed non-exact field strengths G_4 are equivalent to closed non-exact G_4 together with closed non-exact C-field C_3.*

Note that in order to deal with both the equations of motion and the Bianchi identities, we can similarly look at the subcomplex of harmonic forms on Y^{11} with differential d^*, with the obvious changes.

Let $i : M^{10} = \partial Y^{11} \to Y^{11}$ be the inclusion map. Let d_∂, $*_\partial$, d^*_∂ and Δ_∂, respectively, denote the exterior derivative, Hodge star, co-differential and Laplacian on the closed Riemannian ten-manifold M^{10}. The relative cohomology group is defined for smooth p-forms ω whose pullback $i^*\omega$ to M^{10} is zero. Define relative p-forms by $\Omega^p(Y^{11}, \partial Y^{11}) := \{\omega \in \Omega^p(Y^{11}) : i^*\omega = 0\}$, in which the subspaces of closed and exact relative p-forms are given by the conditions $d\omega = 0$ and $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(Y^{11}, \partial Y^{11})$, respectively, and whose quotient is the relative de Rham cohomology.

Boundary conditions on harmonic forms. We would like to pull back harmonic forms on Y^{11} to harmonic forms on the boundary ∂Y^{11}. The simplest question to ask is whether such a pullback is zero. We start on Y^{11} with a C-field C_3, which is closed and co-closed, that is $dC_3 = 0$ and $d^* C_3 = 0$. Then consider the pullback $C'_3 = i^*C_3$. If we take this form to be zero then C_3 itself must be zero. This generalizes to other forms as well, so we get the same conclusion for G_4, G_7 and G_8, using the general results in [7]. Therefore,

Proposition 2 *Let $\mathcal{C}$ stand for any of the fields C_3, G_4, G_7 or G_8 on Y^{11} which is closed and co-closed: $d\mathcal{C} = 0$, $d^*\mathcal{C} = 0$. Let $\mathcal{C}' = i^*\mathcal{C}$ be the pullback of $\mathcal{C}$ to the boundary ∂Y^{11}. If this pullback is zero then the original field is identically zero: $\mathcal{C}' = 0$ implies $\mathcal{C} = 0$. In particular, for a nontrivial G_4 which satisfies the equations of motion and the Bianchi identity in the limit $\ell_p \to 0$, we cannot take the pullback to be zero.*

Therefore, we work with nonzero pullbacks to the boundary.

Dirichlet vs. Neumann forms. The boundary condition $i^*\omega = 0$ appearing in the definition of relative p-forms is called a Dirichlet boundary condition. A form ω satisfying the Dirichlet boundary condition can be thought of as being normal to the boundary; for any $x \in M^{10}$ and $v_1, \cdots, v_p \in T_xY^{11}$, the form $\omega(v_1, \cdots, v_p)$ is not zero only if one of the vectors has a nontrivial component in the direction of the inward-pointing unit normal vector $\mathfrak{i}_n$. For $x \in M^{10}$, let pr: $T_xY^{11} \to T_xM^{10}$ be the orthogonal projection. A p-form ω on Y^{11} is tangent to the boundary if $\omega(v_1, \cdots, v_p) = \omega(\mathrm{pr}(v_1), \cdots, \mathrm{pr}(v_p))$ or, equivalently, if the contraction $\mathfrak{i}_n\omega$ is zero, that is

$$0 = *_\partial \mathfrak{i}_n \omega = i^* * \omega \ . \tag{2.6}$$

This is the Neumann boundary condition. In [25] the authors consider $\mathfrak{i}_n G'_4$ the 3-form that comes from contracting the M-theory G-flux in the bulk Y^{11} with the normal unit vector field to the boundary. Now the Hodge $*$ exchanges Dirichlet forms with Neumann forms, so that if $\omega \in \Omega^p(Y^{11})$ satisfies the Neumann condition then the dual $*\omega \in \Omega^{11-p}(Y^{11})$ satisfies the Dirichlet condition, and vice versa. We immediately have

Proposition 3 *If G_4 (or C_3) is a Neumann form then $*G_4$ (or $*C_3$) is a Dirichlet form and vice versa. Hence the Dirichlet condition cannot be applied both to the C-field and its dual. Consequently, both fields cannot take values in relative cohomology.*

Let the subscripts N and D denote Neumann and Dirichlet boundary conditions respectively. Consider the counterparts of the spaces (2.1), (2.2), and (2.3) in the presence of a boundary

$$\begin{aligned}
\mathcal{E}^p_D(Y^{11}) &:= \{\omega \in \Omega^p(Y^{11}) \;:\; \omega = d\eta \text{ for some } \eta \in \Omega^{p-1}(Y^{11}) \text{ where } i^*\eta = 0\}\;, && (2.7)\\
c\mathcal{E}^p_N(Y^{11}) &:= \{\omega \in \Omega^p(Y^{11}) \;:\; \omega = d^*\xi \text{ for some } \xi \in \Omega^{p+1}(Y^{11}) \text{ where } i^* * \xi = 0\}\;, && (2.8)\\
\mathrm{Harm}^p_D(Y^{11}) &:= \{\omega \in \Omega^p \;:\; d\omega = 0, d^*\omega = 0, \text{ and } i^*\omega = 0\}\;, && (2.9)\\
\mathrm{Harm}^p_N(Y^{11}) &:= \{\omega \in \Omega^p \;:\; d\omega = 0, d^*\omega = 0, \text{ and } i^* * \omega = 0\}\;. && (2.10)
\end{aligned}$$

Note that the boundary conditions apply to the primitive of ω in the first two spaces above, while they apply to ω itself in the last two. If $\omega \in \mathcal{E}^p_D(Y^{11})$ then $\omega = d\eta$ for some Dirichlet form $\eta \in \Omega^{p-1}(Y^{11})$ and $i^*d\eta = d_\partial i^*\eta = 0$, so $\mathcal{E}^p_D(Y^{11})$ is the space of *relatively exact p-forms.* If $\omega \in c\mathcal{E}^p_N(Y^{11})$, then $\omega = d^*\xi$ for some Neumann form $\xi \in \Omega^{p+1}(Y^{11})$ and $i^* * \xi = (-1)^{p+1} i^* d * \xi = (-1)^{p+1} d_\partial i^* * \xi = 0$, so forms in $c\mathcal{E}^p_N(Y^{11})$ satisfy the Neumann boundary condition. The Hodge $*$ operator then takes $\mathrm{Harm}^{11-p}_D(Y^{11})$ to $\mathrm{Harm}^p_N(Y^{11})$.

L^2-decompositions and the Hodge-Morrey-Friedrics (HMF) decomposition. We would like to consider the kinetic terms for the fields (cf. equation (1.2)), and hence we are interested in L^2 (square integrable) expressions. Let $\oplus_{L^2}$ denote orthogonal sum while $\oplus$ denotes direct sum. The Hodge-Morrey-Friedrichs decomposition theorem in our case is the L^2-orthogonal direct sum (see [41])

$$\begin{aligned}
\Omega^p(Y^{11}) &= c\mathcal{E}^p_N(Y^{11}) \oplus_{L^2} \mathrm{Harm}^p_N(Y^{11}) \oplus_{L^2} \mathcal{E}^p(Y^{11}) \cap \mathrm{Harm}^p(Y^{11}) \oplus_{L^2} \mathcal{E}^p_D(Y^{11})\\
&= c\mathcal{E}^p_N(Y^{11}) \oplus_{L^2} c\mathcal{E}^p(Y^{11}) \cap \mathrm{Harm}^p(Y^{11}) \oplus_{L^2} \mathrm{Harm}^p_N(Y^{11}) \oplus_{L^2} \mathcal{E}^p_D(Y^{11})\;, \qquad (2.11)
\end{aligned}$$

with the absolute and relative cohomology groups given by

$$H^p(Y^{11};\mathbb{R}) \cong \mathrm{Harm}^p_N(Y^{11})\;, \qquad H^p(Y^{11}, \partial Y^{11};\mathbb{R}) \cong \mathrm{Harm}^p_D(Y^{11})\;. \qquad (2.12)$$

Applying to the field strength G_4, and denoting the space of these fields by $\{G_4\}$, we get

Proposition 4 (i) *The space of C-fields in M-theory with a boundary, in the limit $\ell_p \to 0$, decomposes into four orthogonal spaces*

$$\{G_4\} = \left\{ \begin{array}{c} G_4 = *dC_6 \\ i^*C_6 = 0 \end{array} \right\} \oplus_{L^2} \left\{ \begin{array}{c} dG_4 = 0 \\ d^*G_4 = 0 \\ i^* * G_4 = 0 \end{array} \right\} \oplus_{L^2} \left\{ \begin{array}{c} G_4 = dC_3 \\ d^*G_4 = 0 \end{array} \right\} \oplus_{L^2} \left\{ \begin{array}{c} G_4 = dC_3 \\ i^*C_3 = 0 \end{array} \right\}$$

(ii) *Denote the above L^2 summands in (i) as type 1, 2, 3, and 4, with corresponding fields $G_4^{(1)}$, $G_4^{(2)}$, $G_4^{(3)}$, and $G_4^{(4)}$, respectively. Then the kinetic term for the C-field decomposes as*

$$\langle G_4, G_4 \rangle_{L^2} = \langle G_4^{(1)}, G_4^{(1)} \rangle_{L^2} + \langle G_4^{(2)}, G_4^{(2)} \rangle_{L^2} + \langle G_4^{(3)}, G_4^{(3)} \rangle_{L^2} + \langle G_4^{(4)}, G_4^{(4)} \rangle_{L^2}\;.$$

Consider the kinetic term for the C-field as in Proposition 4. When Y^{11} has no boundary the variational principle gives $d * G_4 = 0$ for the equation of motion as in (1.1). Now in the presence of a boundary $M^{10} = \partial Y^{11}$, Green's formula gives, for $\alpha \in \Omega^{p-1}(Y^{11})$ and $\beta \in \Omega^p(Y^{11})$, $\langle d\alpha, \beta \rangle_{L^2} - \langle \alpha, d^*\beta \rangle_{L^2} = \int_{M^{10}} i^*\alpha \wedge i^* * \beta$. Therefore, for the C-field we get (we use prime Ξ' for boundary fields)

Lemma 5 *When $\partial Y^{11} \neq \emptyset$, we have $\int_{Y^{11}} dC_3 \wedge *G_4 - \int_{Y^{11}} C_3 \wedge d * G_4 = \int_{\partial Y^{11}} C_3' \wedge G_7'$, where $C_3' := i^*C_3$ and $G_7' := i^* * G_4$.*

Now a more precise statement than the one given in the introduction is the consequence of the Hodge-Morrey-Friedrichs decomposition theorem $\mathrm{Harm}^p_N(Y^{11}) \cap \mathrm{Harm}^p_D(Y^{11}) = \{0\}$. For the C-field this means that in the limit $\ell_p \to 0$

$$\Big\{dG_4 = 0\,, d^* G_4 = 0\,, i^* * G_4 = 0\Big\} \cap \Big\{dG_4 = 0\,, d^* G_4 = 0\,, i^* G_4 = 0\Big\} = \{0\}\,. \tag{2.13}$$

So now we ask whether both of the above spaces can appear in the orthogonal decomposition of a general field strength $G_4 \in \Omega^4(Y^{11})$. A consequence of the the HMF theorem is the DeTurck-Gluck decomposition [13], in which the two outer terms in (2.11) are the same, but the two inner terms are replaced by $\mathcal{E}^p(Y^{11}) \cap c\mathcal{E}^p(Y^{11}) \oplus (\mathrm{Harm}^p_N(Y^{11}) + \mathrm{Harm}^p_D(Y^{11}))$. For G_4 we then have the following alternative to the decomposition in Proposition 4

$$\begin{Bmatrix} G_4 = *dC_6 \\ i^* C_6 = 0 \end{Bmatrix} \oplus_{L^2} \begin{Bmatrix} G_4 = dC_3 \\ G_4 = *dC_6 \end{Bmatrix} \oplus_{L^2} \left[\begin{Bmatrix} dG_4 = 0 \\ d^* G_4 = 0 \\ i^* * G_4 = 0 \end{Bmatrix} \oplus \begin{Bmatrix} dG_4 = 0 \\ d^* G_4 = 0 \\ i^* G_4 = 0 \end{Bmatrix} \right] \oplus_{L^2} \begin{Bmatrix} G_4 = dC_3 \\ i^* C_3 = 0 \end{Bmatrix} \tag{2.14}$$

Here we will have a decomposition of the kinetic term of the C-field as in part (ii) of Proposition 4, taking into account the non-orthogonal direct sum in (2.14).

The reason for the non-orthongonality, as explained more generally in [41], is the fact that some of the cohomology of Y^{11} comes from the interior of Y^{11} and some comes from the boundary ∂Y^{11}. First, in absolute cohomology the interior part is $\ker(i^* : H^p(Y^{11}; \mathbb{R}) \to H^p(\partial Y^{11}; \mathbb{R}))$. This is the condition imposed in [14] (see the introduction). Since $H^p(Y^{11}; \mathbb{R}) \cong \mathrm{Harm}^p(Y^{11})$ then the boundary portion is the subspace of the harmonic Neumann fields which pull back to zero in the cohomology of the boundary,

$$\mathcal{E}^p_\partial(\partial Y^{11}) \cap \mathrm{Harm}^p_N(Y^{11}) := \big\{\omega \in \mathrm{Harm}^p_N(Y^{11}) \ : \ i^*\omega = d\varphi \text{ for some } \varphi \in \Omega^{p-1}(\partial Y^{11})\big\}\ . \tag{2.15}$$

For the field strength G_4 of the C-field we have

$$\mathcal{E}^4_\partial(\partial Y^{11}) \cap \mathrm{Harm}^4_N(Y^{11}) := \Big\{G_4 \mid dG_4 = 0\,, d^* G_4 = 0\,, i^* * G_4 = 0 \ : \ i^* G_4 = dC_3' \text{ for some } C_3' \in \Omega^3(\partial Y^{11})\Big\}\ .$$

Second, for relative cohomology, let $j : Y^{11} = (Y^{11}, \emptyset) \to (Y^{11}, \partial Y^{11})$ be the inclusion. The Hodge $*$ operator exchanges the space $\mathrm{Harm}^4_D(Y^{11})$ with $\mathrm{Harm}^7_N(Y^{11})$ and $\mathrm{Harm}^7_D(Y^{11})$ with $\mathrm{Harm}^4_N(Y^{11})$. The Hodge $*$ also exchanges the boundary subspace $\mathcal{E}^7(Y^{11}) \cap \mathrm{Harm}^7_D(Y^{11})$ with the boundary subspace $c\mathcal{E}^4(Y^{11}) \cap \mathrm{Harm}^4_N(Y^{11})$. For the C-field, we have the effect as the exchange

$$\begin{Bmatrix} G_4 = dC_3, & dG_4 = 0 \\ d^* G_4 = 0, & i^* * G_4 = 0 \end{Bmatrix} \stackrel{*}{\longleftrightarrow} \begin{Bmatrix} G_7 = *dC_3, & dG_7 = 0 \\ d^* G_7 = 0, & i^* G_7 = 0 \end{Bmatrix}\ . \tag{2.16}$$

Similar statements can be deduced for G_4 replaced with G_7 and C_3 replaced with C_6. The portion in relative cohomology coming from the boundary is formed out of those Dirichlet fields which are exact, $\mathcal{E}^p(Y^{11}) \cap \mathrm{Harm}^p_D(Y^{11})$, while the portion coming from the interior is the subspace $c\mathcal{E}^p_\partial(\partial Y^{11}) \cap \mathrm{Harm}^p_D = \big\{\omega \in \mathrm{Harm}^p_D(Y^{11}) \ : \ i^* * \omega = d\psi\big\}$ for some $\psi \in \Omega^{10-p}(\partial Y^{11})$. This the same as the space on the left in (2.16) except that $C_3 \in \Omega^3(Y^{11})$ is replaced with $C_3' \in \Omega^3(\partial Y^{11})$. So the spaces $\mathrm{Harm}^p_N(Y^{11})$ and $\mathrm{Harm}^p_D(Y^{11})$ admit the L^2-orthogonal decomposition into interior and boundary subspaces [13]

$$\begin{aligned} \mathrm{Harm}^4_D(Y^{11}) &= \mathcal{E}^4(Y^{11}) \cap \mathrm{Harm}^4_D(Y^{11}) \oplus c\mathcal{E}^4_\partial(\partial Y^{11}) \cap \mathrm{Harm}^4_D(Y^{11})\,, \\ \mathrm{Harm}^4_N(Y^{11}) &= c\mathcal{E}^4(Y^{11}) \cap \mathrm{Harm}^4_N(Y^{11}) \oplus \mathcal{E}^4_\partial(\partial Y^{11}) \cap \mathrm{Harm}^4_N(Y^{11})\,. \end{aligned}$$

For the C-field the first of the two expressions gives the L^2-orthogonal decomposition

$$\begin{Bmatrix} G_4 & dG_4 = 0 \\ d^* G_4 = 0, & i^* G_4 = 0 \end{Bmatrix} = \begin{Bmatrix} G_4 = dC_3, & dG_4 = 0 \\ d^* G_4 = 0, & i^* G_4 = 0 \end{Bmatrix} \oplus_{L^2} \begin{Bmatrix} G_4, & dG_4 = 0 \\ d^* G_4 = 0, & i^* G_4 = dC_3',\ C_3' \in \Omega^3(\partial Y^{11}) \end{Bmatrix}\,, \tag{2.17}$$

while the second gives

$$\begin{Bmatrix} G_4 & dG_4=0 \\ d^* G_4=0, & i^* * G_4=0 \end{Bmatrix} = \begin{Bmatrix} G_4=*dC_6, & dG_4=0 \\ d^* G_4=0, & i^* * G_4=0 \end{Bmatrix} \oplus_{L^2} \begin{Bmatrix} G_4, & dG_4=0 \\ d^* G_4=0, & i^* * G_4 = dC_6',\ C_6' \in \Omega^6(\partial Y^{11}) \end{Bmatrix} . \tag{2.18}$$

Let us illustrate equation (2.18); write the two summands on the right hand side as $Q \oplus R$. Letting $d^* \xi_5$ (or $*dC_6$) $\in Q$ and $\tilde{G}_4 \in R$ we have $i^* \tilde{G}_4 = dC_3'$ for some $C_3' \in \Omega^3(\partial Y^{11})$. Then

$$\begin{aligned} \langle \tilde{G}_4, d^* \xi_5 \rangle_{L^2} &= \langle d\tilde{G}_4, \xi_5 \rangle_{L^2} - \int_{\partial Y^{11}} i^* \tilde{G}_4 \wedge i^* * \xi_5 \\ &= -\int_{\partial Y^{11}} dC_3' \wedge i^* * \xi_5 \\ &= \langle d\tilde{C}_3, d^* \xi_5 \rangle_{L^2} && \tilde{C}_3 \in \Omega^3(Y^{11}) \text{ extension of } C_3' \text{ to } Y^{11} \\ &= \langle \tilde{C}_3, d^* d^* \xi_5 \rangle_{L^2} + \int_{Y^{11}} i^* \tilde{C}_3 \wedge i^* * d^* \xi_5 && \text{Green's Theorem} \\ &= 0 && i^* * d^* \xi_5 = 0 \text{ since } d^* \xi_5 \text{ is Neumann.} \end{aligned}$$

Write the decompositions (2.17) and (2.18) schematically as

$$\{G_4^D\} = \{G_4^{D,e}\} \oplus_{L^2} \{G_4^{D,c}\} , \qquad \{G_4^N\} = \{G_4^{N,c}\} \oplus_{L^2} \{G_4^{N,e}\} , \tag{2.19}$$

where the superscripts D and N refer to Dirichlet and Neumann, while the extra superscripts e and c refer to exact and co-exact, respectively. We summarize the above discussion with

Proposition 6 *The field strength splits into Dirichlet and Neumann forms G_4^D and G_4^N, whose L^2-inner product decomposes according to Dirichlet and Neumann boundary conditions and to exactness and co-exactness on the boundary as*

$$\begin{aligned} \langle G_4^D, G_4^D \rangle_{L^2} &= \langle G_4^{D,e}, G_4^{D,e} \rangle_{L^2} \oplus_{L^2} \langle G_4^{D,c}, G_4^{D,c} \rangle_{L^2} , \\ \langle G_4^N, G_4^N \rangle_{L^2} &= \langle G_4^{N,e}, G_4^{N,e} \rangle_{L^2} \oplus_{L^2} \langle G_4^{N,c}, G_4^{N,c} \rangle_{L^2} . \end{aligned}$$

This is a refinement of the split of G_4 into G_4^D and G_4^N, for instance in [25]. We will consider "mixing" in section 2.3.

Integral forms. The partition function of the C-field requires integrating over the space of harmonic forms as well as summing over torsion fields. We have considered the former in a lot of detail so far, so we now provide remarks about including the latter. Denote by $\mathrm{Harm}^p_{N,\mathbb{Z}}(M)$ the image in $\mathrm{Harm}^p_N(M)$ of the integer lattice $H^p_{\mathbb{Z}}(M;\mathbb{R})$ of $H^p(M;\mathbb{R})$ under the isomorphism $H^p(M;\mathbb{R}) \cong \mathrm{Harm}^p_N(M)$. A form α is in $\mathrm{Harm}^p_{N,\mathbb{Z}}(M)$ if and only if $\int_S \alpha \in \mathbb{Z}$ for any singular p-cycle S of M. Similarly, Denote by $\mathrm{Harm}^p_{D,\mathbb{Z}}(M)$ the image in $\mathrm{Harm}^p_D(M)$ of the integer lattice $H^p_{\mathbb{Z}}(M, \partial M;\mathbb{R})$ of $H^p(M, \partial M;\mathbb{R})$ under the isomorphism $H^p(M, \partial M;\mathbb{R}) \cong \mathrm{Harm}^p_D(M)$. A form α is in $\mathrm{Harm}^p_{D,\mathbb{Z}}(M)$ if and only if $\int_S \alpha \in \mathbb{Z}$ for any relative singular p-cycle S of M. With these notions, the above discussions can be extended (but we do not need that explicitly here).

2.2 Duality on the boundary and boundary value problems

We seek to characterize the resulting duality between the fields pulled back to the boundary ∂Y^{11}, starting with Hodge duality on Y^{11}. The expression in Lemma 5 suggests that we look at C_3' and G_7'.

Dirichlet-to-Neumann map on the forms. There is an operator that takes care of Hodge duality on the boundary and which treats the field and its dual at the same time [23] [42]. Define the Dirichlet-to-Neumann (D-to-N) map $\Lambda_p : \Omega^p(\partial Y^{11}) \to \Omega^{10-p}(\partial Y^{11})$ as follows [4] [23] [40]. If $\varphi \in \Omega^p(\partial Y^{11})$ is a smooth p-form on the boundary the define $\Lambda_p\varphi := i^* * d\omega$, which is independent of the choice of ω due to the presence of the exterior derivative. Hence for the boundary fields C_3' and G_7' we have $\Lambda_3 C_3' := i^* * dC_3$ and $\Lambda_7 G_7' := i^* * dG_7$ with C_3 and G_7 a three- and seven-form, respectively, on Y^{11}. Then the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad \text{and} \quad i^* d^* \omega = 0 \tag{2.20}$$

has a unique solution up to the addition of an arbitrary harmonic Dirichlet field $\lambda \in \text{Harm}_D^p(Y^{11})$ [40]. From [4], $d\omega \in \text{Harm}^{p+1}(Y^{11})$ and $d^*\omega = 0$, so that (2.20) is equivalent to the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad \text{and} \quad d^* \omega = 0\,. \tag{2.21}$$

We then have, in our case, the following two BVPs for the two boundary fields C_3' and G_7'

$$\begin{aligned} \text{BVP1} \quad &: \quad \Delta C_3 = 0\,, \qquad i^*C_3 = C_3'\,, \qquad d^* C_3 = 0\,, \qquad (2.22)\\ \text{BVP2} \quad &: \quad \Delta G_7 = 0\,, \qquad i^*G_7 = G_7'\,, \qquad d^* G_7 = 0\,. \qquad (2.23)\end{aligned}$$

The kernel and image of the D-to-N operator are given by $i^*\text{Harm}^p(Y^{11}) = \text{ker}\Lambda_p = \text{im}\Lambda_{10-p}$, which gives, for a 3-form, $i^*\text{Harm}^3(Y^{11}) = \text{ker}\Lambda_3 = \text{im}\Lambda_7$. In fact, from [41], this kernel has a direct sum decomposition $\text{ker}\Lambda_p = i^*\text{Harm}^p(Y^{11}) = i^*c\mathcal{E}^p(Y^{11}) \cap \text{Harm}^p(Y^{11}) + \mathcal{E}^p(\partial Y^{11})$, so that $\text{ker}\Lambda_p/\mathcal{E}^p(\partial Y^{11}) \cong c\mathcal{E}^p(Y^{11}) \cap \text{Harm}^p(Y^{11})$, with dimension equal to that of the boundary subspace of $H^p(Y^{11};\mathbb{R})$. For our two fields on the boundary we have

$$\begin{aligned} \text{ker}\Lambda_3 &= \left\{C_3' = i^*C_3 \mid C_3 = d^* G_4\,,\ dC_3 = 0\,,\ d^* C_3 = 0\right\} \oplus \{C_3' = dB_2'\}\,, \qquad (2.24)\\ \text{ker}\Lambda_7 &= \left\{G_7' = i^*G_7 \mid G_7 = d^* G_8\,,\ dG_7 = 0\,,\ d^* G_7 = 0\right\} \oplus \{G_7' = dB_6'\}\,. \qquad (2.25)\end{aligned}$$

This D-to-N operator can treat the field and its dual in a unified way by considering

$$\Pi : \Omega^k(\partial Y^{11}) \times \Omega^{11-k}(\partial Y^{11}) \longrightarrow \Omega^{10-k}(\partial Y^{11}) \times \Omega^{k-1}(\partial Y^{11}) \tag{2.26}$$

defined by $\Pi\binom{\varphi}{\psi} = \binom{i^* * d\omega}{i^* d^* \omega}$, where $\omega \in \Omega^k(Y^{11})$ is the solution of the boundary value problem $\{\Delta\omega = 0, i^*\omega = \varphi, i^* * \omega = \psi\}$. This is best described by splitting into two linear operators (Φ, Ψ) [42] as $\Pi = \begin{pmatrix} \Phi & , & (-1)^k\Psi \\ \Psi & , & (-1)^{k+1}\Phi \end{pmatrix}$ on $\Omega^k(\partial Y^{11}) \times \Omega^{11-k}(\partial Y^{11})$, with $\Phi : \Omega^k(\partial Y^{11}) \to \Omega^{10-k}(\partial Y^{11})$ and $\Psi : \Omega^k(\partial Y^{11}) \to \Omega^{k-1}(\partial Y^{11})$ defined by the expressions $\Phi\varphi = i^* * d\omega$ and $\Psi\varphi = i^* d^* \omega$, where $\omega \in \Omega^k(Y^{11})$ is now a solution to the boundary value problem $\{\Delta\omega = 0, i^*\omega = \varphi, i^* * \omega = 0\}$. We are interested in the pair $k = (3, 7)$, which is equivalent to the pair $k = (4, 8)$ – the effect will be simply an exchange of factors; for instance, for $k = 3$ we will have $\Pi : \Omega^3(\partial Y^{11}) \times \Omega^8(\partial Y^{11}) \to \Omega^7(\partial Y^{11}) \times \Omega^2(\partial Y^{11})$, while for $k = 8$ we have $\Pi : \Omega^8(\partial Y^{11}) \times \Omega^3(\partial Y^{11}) \to \Omega^2(\partial Y^{11}) \times \Omega^7(\partial Y^{11})$.

Let $b_k(Y^{11}) = \text{dim}H^k(Y^{11};\mathbb{R})$ be the kth Betti number of Y^{11}. Then, from [42], $b_k(Y^{11}) = \dim(\ker(\Phi))$ and the kernel of the operator $\Phi_k : \Omega^k(\partial Y^{11}) \to \Omega^{10-k}(\partial Y^{11})$ consists of the boundary traces of harmonic Neumann fields, i.e. $\ker(\Phi_k) = i^*\text{Harm}_N^k(Y^{11})$. For C_3' and G_7' we have

Theorem 7 *(i) The (solvable) boundary value problem which involves duality on the boundary is given by*

$$\Pi\begin{pmatrix} C_3' \\ G_8' \end{pmatrix} = \begin{pmatrix} i^* * dC_3 \\ i^* d^* C_3 \end{pmatrix}, \Phi_3 C_3' = i^* * dC_3, \Psi_3 C_3' = i^* d^* C_3, C_3 \text{ is a solution to BVP}$$
$$\{\Delta C_3 = 0, i^*C_3 = C_3', i^* * C_3 = G_8\}$$

and similarly for G_7 with $\Phi_7 G_7' = i^* * dG_7$ and $\Psi_7 G_7' = i^* d^* G_7$.
(ii) The Betti numbers of Y^{11} are given in terms of the D-to-N operators as

$$\begin{aligned} b_3(Y^{11}) &= \dim\ker(\Phi_3) = \dim(i^*\text{Harm}_N^3(Y^{11})) = \dim\{C_3' \mid C_3' = i^*C_3, dC_3 = 0, d^* C_3 = 0, i^* * C_3 = 0\}\,,\\ b_7(Y^{11}) &= \dim\ker(\Phi_7) = \dim(i^*\text{Harm}_N^7(Y^{11})) = \dim\{G_7' \mid G_7' = i^*G_7, dG_7 = 0, d^* G_7 = 0, i^* * G_7 = 0\}\,.\end{aligned}$$

2.3 Mixing Dirichlet with Neumann and Poincaré duality angles

We have seen how the absolute harmonic forms and the relative harmonic forms have no elements in common except for 0. We have also seen that the interior parts of the absolute and the relative cohomology have elements in common. We now characterize these elements making use of results from [41]. To that end, we consider the Poincaré duality angles, which are the principal angles between the following two interior subspaces (cf. equation (2.15))

$$\mathcal{A} := \mathcal{E}^3_\partial(\partial Y^{11}) \cap \mathrm{Harm}^3_N(Y^{11}) = \Big\{ C_3 \mid dC_3 = 0, d^* C_3 = 0, i^* * C_3 = 0 \ : \ i^* C_3 = dB_2' \text{ for } B_2' \in \Omega^2(\partial Y^{11}) \Big\} \ ,$$
$$\mathcal{B} := c\mathcal{E}^3_\partial(\partial Y^{11}) \cap \mathrm{Harm}^3_D(Y^{11}) = \Big\{ C_3 \mid dC_3 = 0, d^* C_3 = 0, i^* C_3 = 0 \ : \ i^* * C_3 = dG_7' \text{ for } G_7' \in \Omega^7(\partial Y^{11}) \Big\} \ .$$

Following the general construction in [41], let $\mathsf{proj}_D : \mathrm{Harm}^3_N(Y^{11}) \to \mathrm{Harm}^3_D(Y^{11})$ be the orthogonal projection onto the space of Dirichlet fields. This takes a closed and co-closed C_3 whose Hodge dual pulls back to zero on the boundary to one which is closed and co-closed and itself pulls back to zero on the boundary. Then $\mathsf{proj}_D \mathcal{A} = \mathcal{B}$. The nonzero singular values of proj_D, that is the square roots of the eigenvalues of the nonnegative adjoint operator $\mathsf{proj}_D^* \circ \mathsf{proj}_D$, are given by $\cos\theta$ and so define the Poincaré duality angle θ. This is also given by the nonzero singular values of $\mathsf{proj}_N : \mathrm{Harm}^3_D(Y^{11}) \to \mathrm{Harm}^3_N(Y^{11})$ onto the space of Neumann fields. Then $\cos^2\theta$ are the nonzero eigenvalues of the compositions $\mathsf{proj}_N \circ \mathsf{proj}_D : \mathrm{Harm}^3_N(Y^{11}) \to \mathrm{Harm}^3_N(Y^{11})$ and $\mathsf{proj}_D \circ \mathsf{proj}_N : \mathrm{Harm}^3_D(Y^{11}) \to \mathrm{Harm}^3_D(Y^{11})$.

Let T denote the Hilbert transform [4] defined as $T := d_\partial \Lambda^{-1}$ and which is well-defined on the subset of forms on ∂Y^{11} given by $i^*\mathrm{Harm}^p(Y^{11}) = \mathrm{im}\Lambda_7$. Let $C_3^N \in \mathrm{Harm}^3_N(Y^{11})$ and $\mathsf{proj}_D C_3^N = C_3^D \in \mathrm{Harm}^3_D(Y^{11})$ be the orthogonal projection of C_3^N onto $\mathrm{Harm}^3_D(Y^{11})$. On the other hand, let $C_3^D \in \mathrm{Harm}^3_D(Y^{11})$ and $\mathsf{proj}_N C_3^D = C_3^N \in \mathrm{Harm}^3_N(Y^{11})$ be the orthogonal projection of C_3^D onto $\mathrm{Harm}^3_N(Y^{11})$. Then, using [41],

$$T i^* C_3^N = i^* * C_3^D \ , \qquad T i^* * C_3^D = -i^* C_3^N \ . \tag{2.27}$$

There is connection between Dirichlet-to-Neumann map and the Poincaré duality angles. Consider the restriction $\widetilde{T}$ of the Hilbert transform T to the pullback of Neumann harmonic three-forms

$$i^*\mathrm{Harm}^3_N(Y^{11}) = \{ C_3' \mid C_3' = i^* C_3, dC_3 = 0, d^* C_3 = 0, i^* * C_3 = 0 \} \ . \tag{2.28}$$

Then [41] the quantities $-\cos^2\theta$ are the nonzero eigenvalues of $\widetilde{T}^2$. We can similarly consider G_4 in place of C_3 for which similar results as above hold.

Proposition 8 *(i) Let $G_4^N \in \mathrm{Harm}^4_N(Y^{11})$ and $\mathsf{proj}_D G_4^N = G_4^D \in \mathrm{Harm}^4_D(Y^{11})$ be the orthogonal projection onto $\mathrm{Harm}^4_D(Y^{11})$. Then the Hilbert transform acts as $T i^* G_4^N = -i^* * G_4^D$, and $T i^* * G_4^D = i^* G_4^N$.*
(ii) $\langle G_4^N \ , \ G_4^D \rangle_{L^2} = ||G_4^N||_{L^2} \ ||G_4^D||_{L^2} \cos\theta$, where $\cos^2\theta$ is the eigenvalue of the composition $\mathsf{proj}_D \circ \mathsf{proj}_N : \mathrm{Harm}^4_D(Y^{11}) \to \mathrm{Harm}^4_D(Y^{11})$.

This is the "mixing" which complements Proposition 6. We now illustrate with examples in M-theory.

Disk bundles over ten-manifolds. Consider the complex six-dimensional projective space $\mathbb{C}P^6$. Define a one-parameter family of compact Riemannian manifolds with boundary $Z_r^{12} := \mathbb{C}P^6 - \mathbb{B}_r(x)$, where $\mathbb{B}_r(x)$ is an open ball of radius $r \in (0, \pi/2)$ centered at a point x in $\mathbb{C}P^6$. The nontrivial cohomology groups of Z_r^{12} are

$$H^{2k}(Z_r^{12}; \mathbb{R}) \cong H^{12-2k}(Z_r^{12}, \partial Z_r^{12}; \mathbb{R}) = \mathbb{R}; \quad k = 0, \cdots, 5. \tag{2.29}$$

The twelve-manifold Z_r^{12} is in fact a 2-disk bundle over $\mathbb{C}P^5$, and the boundary ∂Z_r^{12} is homeomorphic to the eleven-sphere S^{11}. For each $1 \le k \le 5$, harmonic $2k$-fields satisfying Neumann and Dirichlet boundary conditions can be constructed [41]. Let $h_t : S^{11} \to S_t^{11}$ the diffeomorphism of the unit sphere with the hypersurface at constant distance t from $\mathbb{C}P^5$. Let $H : S^{11} \to \mathbb{C}P^5$ be the Hopf fibration and let v be the vector field on $\mathbb{C}P^6$ which restricts on each hypersurface to the pushforward by h_t of the unit vector field in

the Hopf direction on S^{11}. Define α to be the dual to v, and let $\tau = dt$ be the 1-form dual to ∂/∂_t and define η to be the two-form which restricts on each S^{11}_t to $(H \circ h_t)^* \eta_{\mathbb{C}P^5}$, where $\eta_{\mathbb{C}P^5}$ is the standard symplectic form on $\mathbb{C}P^5$. Away from $\mathbb{C}P^5$ the manifold Z^{12}_r is topologically a product $S^{11} \times I$, so exterior derivatives can be computed as in S^{11}. Thus $d\alpha = -2\eta$ and η and τ are closed. Closed and co-closed 4-forms G_4^N and G_4^D satisfying Neumann and Dirichlet conditions, respectively

$$G_4^N := f_N(t)\eta^2 + g_N(t)\alpha \wedge \eta \wedge \tau \,, \qquad G_4^D := f_D(t)\eta^2 + g_D(t)\alpha \wedge \eta \wedge \tau \,, \tag{2.30}$$

can be constructed for functions f and g. The angle θ between G_4^N and G_4^D is in Proposition 8 with θ given by [41] $\cos\theta = (1 - \sin^6 r)/[(1 + \sin^6 r)^2 + \frac{1}{2}\sin^6 r]^{1/2}$. The same result holds for the case when the Euler class of the two-disk bundle over $\mathbb{C}P^5$ is varied. The $\mathbb{D}^2$ bundle with Euler class m over $\mathbb{C}P^5$ has boundary $L(m,1)$, the lens space which is the quotient of the sphere S^{11} by the action of $\mathbb{Z}_m$ given by $e^{2\pi i/m} \cdot (z_0, \cdots, z_6) = \left(e^{2\pi i/m} z_0, \cdots, e^{2\pi i/m} z_6\right)$. The result for the angle is the same as above and is independent of the Euler class m. As $r \to 0$, $\theta \to 0$ so that the two forms are orthogonal in this case.

2.4 Effect of including the M5-brane

Consider the M5-brane worldvolume W^6 embedded in eleven-dimensional spacetime $\iota : W^6 \to Y^{11}$. Take $W^6 = S^5 \times I$ and $Y^{11} = M^{10} \times I$, where I corresponds to compact time direction (we could also use $\mathbb{R}$ in place of the interval I). We identify a tubular neighborhood of S^5 in M^{10} with the total space of the normal bundle $N \to S^5$. The unit sphere bundle of radius r , $X^9 = S_r(N)$ is an associated S^4 bundle $\pi : X^9 \to S^5$. Now we can construct in ten dimensions, in analogy to what was done in eleven dimensions in [14], a ten-manifold M_r^{10} with boundary X^9 by removing the disk bundle of radius r, $M_r^{10} = M^{10} - \mathbb{D}(N)$.

Example: Grassmannians. Consider the Grassmannian of oriented two-planes in Euclidean space $\mathbb{R}^7$, $\mathrm{Gr}_2\mathbb{R}^7 = SO(7)/SO(5)$, which is a ten-manifold with Riemannian submersion metric induced by the bi-invariant metrics on the Lie groups $SO(7)$ and $SO(5)$. This has a subGrassmannian $\mathrm{Gr}_1\mathbb{R}^6 = SO(6)/SO(5)$, the Grassmannian of oriented lines in $\mathbb{R}^6$, which is just the unit five-sphere S^5. Consider the one-parameter family of manifolds $M_r^{10} := \mathrm{Gr}_2\mathbb{R}^7 - \nu_r(\mathrm{Gr}_1\mathbb{R}^6)$, where $\nu_r(\mathrm{Gr}_1\mathbb{R}^6)$ is the open tubular neighborhood of radius r around $\mathrm{Gr}_1\mathbb{R}^6$, which is topologically the unit tangent bundle of S^5. M_r^{10} is a $\mathbb{D}^2$ bundle over the eight-manifold $\mathrm{Gr}_2\mathbb{R}^6 = SO(6)/SO(4)$. The boundary ∂M_r^{10} is homeomorphic to the unit tangent bundle US^5. This has the same rational cohomology as $S^5 \times S^4$, so all of this cohomology is interior except $H^4(M_r^{10};\mathbb{R})$ and so $H^4(M_r^{10}, \partial M_r^{10};\mathbb{R})$ could potentially have a 1-dimensional boundary subspace. The Poincaré duality angle θ between the concrete realizations of $H^4(M_r^{10};\mathbb{R})$ and $H^4(M_r^{10}, \partial M_r^{10};\mathbb{R})$ is given by [41] $\cos\theta = (1 - \sin^5 r)/[(1 + \sin^5 r)^2 + \frac{1}{6}\sin^5 r]^{1/2}$. Again $\theta \to 0$ as $r \to 0$ so that the forms become orthogonal.

Proposition 9 *Even on a closed eleven-manfold, there is a mixing between Dirichlet and Neumann field strengths G_4 (or C-fields) due to the presence of the M5-brane worldvolume*

We now consider general six- and eleven-manifolds. Let Y^{11} be a closed smooth oriented Riemannian eleven-manifold and let M^6 be a closed submanifold of codimension five representing the worldvolume of the M5-brane. Define the compact Riemannian eleven-manifold $Y_r^{11} := Y^{11} - \nu_r(M^6)$, where $\nu_r(M^6)$ is the open tubular neighborhood of radius r about M^6. The radius is taken to small enough so that the ten-dimensional boundary ∂Y_r^{11} is smooth. Let θ be the Poincaré duality angle of Y_r^{11} in dimension 4. We are interested in the behavior of θ as $r \to 0$. Examples show that $\theta \sim \mathcal{O}(r^m)$, where $m > 0$. In fact in [41] this is conjectured to hold in general with m the codimension of M in Y, being five in our case. The physical counterpart of this conjecture is then

Conjecture 1 *For a general M5-brane worldvolume in a general eleven-dimensional manifold, the mixing between $H^4(M_r^{10};\mathbb{R})$ and so $H^4(M_r^{10}, \partial M_r^{10};\mathbb{R})$ is nonzero in general, and vanishes in the limit when the size of the tubular is very small. Thus for a macroscopic M5-brane the effect of the Poincaré duality angle is visible, and the effect disappears in the microscopic limit.*

3 Global aspects: Bundles and the phase of the partition function

We have so far considered local questions related to differential forms and corresponding differential equations. In this section we shift to more global aspects, that is to bundles and to integrals of differential forms. We first consider the E_8 bundle and then consider the phase of the partition function.

3.1 E_8 gauge theory

In this section we consider the boundary conditions on the bundles involved, especially the E_8 bundle on Y^{11}, and consider conditions and obstructions for extensions. In [45] the case of an abelian gauge theory on a 4-dimensional manifold with a boundary was considered. We generalize to the case of E_8 bundle in eleven dimensions. Although we discuss mainly the eleven-dimensional case, our results apply to other dimensions.

Extension of E_8 bundles. Let (P, A) be a principal bundle P with connection $A \in \mathrm{Conn}(P)$ on Y^{11} and (P_∂, A_∂) be a principal bundle on the boundary ∂Y^{11} with the same structure group. Every bundle P on Y^{11} yields by pullback a bundle P_∂ on ∂Y^{11}, i.e. $P_\partial = i^* P$. However the converse is not always true; not every bundle P_∂ on the boundary always the pullback of some bundle P on Y^{11}. We consider the spaces $\mathrm{Prin}(Y^{11})$, $\mathrm{Prin}(\partial Y^{11})$, $\mathrm{Prin}(Y^{11}, \partial Y^{11})$ of principal E_8 bundles over Y^{11}, over ∂Y^{11}, and those over Y^{11} which vanish on the boundary ∂Y^{11}, respectively. Since $BE_8 \sim K(\mathbb{Z}, 4)$ in our range of dimensions we have isomorphisms $\mathrm{Prin}(Y^{11}) \overset{c}{\cong} H^4(Y^{11}; \mathbb{Z})$ and $\mathrm{Prin}(\partial Y^{11}) \overset{c_\partial}{\cong} H^4(\partial Y^{11}; \mathbb{Z})$. Furthermore, we have the following commutative diagram

$$\begin{tikzcd} H^3(\partial Y^{11};\mathbb{Z}) \arrow[r, "\delta"] & H^4(Y^{11},\partial Y^{11};\mathbb{Z}) \arrow[r] & H^4(Y^{11};\mathbb{Z}) \arrow[r, "\iota^*"] & H^4(\partial Y^{11};\mathbb{Z}) \arrow[r, "\delta"] & H^5(Y^{11},\partial Y^{11};\mathbb{Z}) \ . \\ & \mathrm{Prin}(Y^{11},\partial Y^{11}) \arrow[u, "c"] \arrow[r] & \mathrm{Prin}(Y^{11}) \arrow[u, "c"] \arrow[r, "\iota^*"] & \mathrm{Prin}(\partial Y^{11}) \arrow[u, "c_\partial"] & \end{tikzcd} \tag{3.1}$$

The horizontal arrows are exact and the vertical maps are isomorphisms. The first map in the second line associates to every relative bundle the underlying bundle, and the second arrow associates to every bundle P its pullback bundle $P_\partial = \iota^* P$. By exactness in the diagram, a boundary bundle $P_\partial \in \mathrm{Prin}(\partial Y^{11})$ is the pullback of a bundle $P \in \mathrm{Prin}(Y^{11})$ if and only if

$$\delta(c_\partial(P_\partial)) = 0 \ , \tag{3.2}$$

so that the obstruction to the extendibility of P_∂ is a class of $H^5(Y^{11}, \partial Y^{11}; \mathbb{Z})$. When this is satisfied, P_∂ is in general the pullback of more than one bundle P on Y^{11}, that is P has several extensions to Y^{11}. The extensions are parametrized by the group of relative bundles in a fashion which is one-to-one provided $H^3(\partial Y^{11}; \mathbb{Z}) = 0$.

Extensions of gauge transformations. Let $\mathcal{G}$ denote the group of gauge transformations [2] of the E_8 bundle. Starting with a gauge transformation $U \in \mathcal{G}(\partial Y^{11})$ we get by pullback a gauge transformation $U_\partial \in \mathcal{G}(\partial Y^{11})$, that is, $U_\partial = i^* U$. The converse is not true, that is not every boundary gauge transformation U_∂ the pullback of some gauge transformation U. In [45] the obstruction to such an extension was identified for the abelian case in four dimensions, and so here we work out the analog for the E_8 gauge theory in eleven dimensions (following similar arguments). Consider the gauge transformation class groups

$$\mathcal{C}(Y^{11}) = \mathcal{G}(Y^{11})/\mathcal{G}_c(Y^{11}) \ , \quad \mathcal{C}(\partial Y^{11}) = \mathcal{G}(\partial Y^{11})/\mathcal{G}_c(\partial Y^{11}) \ , \quad \mathcal{C}(Y^{11}, \partial Y^{11}) = \mathcal{G}(Y^{11}, \partial Y^{11})/\mathcal{G}_c(Y^{11}, \partial Y^{11}) \ ,$$

[2] i.e. gauge group for a mathematician.

where the factor groups are connected components to the corresponding identity gauge transformations. Now we have an analog of the diagram (3.1),

$$\begin{tikzcd} H^2(\partial Y^{11};\mathbb{Z}) \arrow[r, "\delta"] & H^3(Y^{11},\partial Y^{11};\mathbb{Z}) \arrow[r] & H^3(Y^{11};\mathbb{Z}) \arrow[r, "\iota^*"] & H^3(\partial Y^{11};\mathbb{Z}) \arrow[r, "\delta"] & H^4(Y^{11},\partial Y^{11};\mathbb{Z}) \\ & \mathcal{C}(Y^{11},\partial Y^{11}) \arrow[u, "q"] \arrow[r] & \mathcal{C}(Y^{11}) \arrow[u, "q"] \arrow[r, "\iota^*"] & \mathcal{C}(\partial Y^{11}) \arrow[u, "q_\partial"] & \end{tikzcd} \quad . \tag{3.3}$$

Here q assigns to each gauge transformation U its characteristic class $q(U)$. By exactness, a gauge transformation class $[U_\partial] \in \mathcal{C}(\partial Y^{11})$ is the pullback of a gauge transformation class $[U] \in \mathcal{C}(Y^{11})$ if an only if

$$\delta(q_\partial([U_\partial])) = 0 \ . \tag{3.4}$$

Hence the obstruction to the extendibility of $[U_\partial]$ is a class of $H^4(Y^{11},\partial Y^{11};\mathbb{Z})$. When $[U_\partial]$ satisfies (3.4), $[U_\partial]$ is the pullback of possibly several gauge transformation classes $[U] \in \mathcal{C}(Y^{11})$, that is $[U_\partial]$ has possibly several extensions to Y^{11}. These extensions are parametrized by the group of relative gauge transformations $\mathcal{G}(Y^{11},\partial Y^{11})$, which is a one-to-one correspondence when $H^2(\partial Y^{11};\mathbb{Z}) = 0$.

The boundary condition on these bundles is $i^*P = P_\partial$. Also, a natural choice for the connection is

$$i^*A = A_\partial \ , \tag{3.5}$$

where $A_\partial \in \mathrm{Conn}(P_\partial)$ is a fixed connection. These boundary conditions are not preserved under the action of gauge transformations $\mathcal{G}(Y^{11})$ on Y^{11}. However, they are preserved by the group of relative gauge transformations $\mathcal{G}(Y^{11},\partial Y^{11})$. The allowed variations on the connection $A \in \mathrm{Conn}(P)$ preserving the boundary condition (3.5) are Lie algebra-valued 1-forms $\delta_g A \in \Omega^1(Y^{11}) \otimes \mathfrak{e}_8$ such that $\iota^*\delta_g A = 0$, that is $\delta_g A \in \Omega^1_D(Y^{11}) \otimes \mathfrak{e}_8$ satisfies the Dirichlet boundary conditions. Our previous constructions can be extended to the nonabelian case. Define $d^A_\partial = d_\partial + A_\partial$, and take d^{*A}_∂ be its adjoint. For instance, the *nonabelian Hilbert transform* will now become $T_A := d^A_\partial \Lambda^{-1} = T + A_\partial \Lambda^{-1}$, and similarly for other entities.

We summarize our findings in this section in

Theorem 10 *(i) An E_8 bundle $P_\partial \in \mathrm{Prin}(\partial Y^{11})$ is the pullback of a bundle $P \in \mathrm{Prin}(Y^{11})$ if and only if $\delta(c_\partial(P_\partial)) = 0$, so that the obstruction to the extendibility of P_∂ is a class of $H^5(Y^{11},\partial Y^{11};\mathbb{Z})$. When this is satisfied, the extensions are parametrized by the group of relative bundles in a fashion which is one-to-one provided $H^3(\partial Y^{11};\mathbb{Z}) = 0$.*
(ii) A gauge transformation class $[U_\partial] \in \mathcal{C}(\partial Y^{11})$ is the pullback of a gauge transformation class $[U] \in \mathcal{C}(Y^{11})$ if an only if $\delta(q_\partial([U_\partial])) = 0$, so that the obstruction to the extendibility of $[U_\partial]$ is a class of $H^4(Y^{11},\partial Y^{11};\mathbb{Z})$. When this is satisfied, the extensions are parametrized by the group of relative gauge transformations $\mathcal{G}(Y^{11},\partial Y^{11})$, which is a one-to-one correspondence when $H^2(\partial Y^{11};\mathbb{Z}) = 0$.

3.2 The phase of the partition function via the adiabatic limit

In this section we consider the effect of having the boundary ∂Y^{11} on the phase of the partition function, given by the exponentiated eta-invariants [15] (cf. equation (1.2))

$$\Phi(C_3) = \exp\left[2\pi i \left(\frac{1}{2}\overline{\eta}_{E_8} + \frac{1}{4}\overline{\eta}_{RS}\right)\right] \ , \tag{3.6}$$

where $\overline{\eta} := \frac{1}{2}(\eta + h)$, each for an E_8 bundle and the Rarita-Schwinger bundle, and h the number of zero modes of the corresponding twisted Dirac operator. In [38] this is written in terms of the eta invariant corresponding to the signature operator on Y^{11}. Thus, the phase can be studied either using Dirac operators or the signature operator (the latter can in a sense also be considered a Dirac operator). The phase (3.6)

is the result of the use of the APS index theorem for a twelve-manifold Z^{12} with a boundary $\partial Z^{12} = Y^{11}$ on the phase in twelve dimensions written in terms the Atiyah-Singer index [44]. The dimensional reduction to ten-dimensions is performed in [31] [35] [38] via the adiabatic limit of the eta invariants which result in expressions for the phase in ten dimensions.

We would like to consider the case when Y^{11} has a non-empty boundary ∂Y^{11}. So we no longer assume that Y^{11} is itself a boundary, and the phase (3.6) is taken as a starting point, as in [14]. In M-theory we are in practice interested in eleven-manifolds which are decomposable into a product or which are total spaces of bundles. Therefore, we consider eleven-manifolds with boundary which are products or fiber bundles. Note that the product of a manifold-with-boundary with a manifold without boundary is a manifold with boundary, and similarly for bundles. Therefore, we will consider bundles with total space Y^{11} where either the fiber or the base has a boundary, which makes Y^{11} itself a manifold with boundary.

I. The base space with a boundary. Consider a bundle $N^n \hookrightarrow Y^{11} \xrightarrow{\pi} X^{11-n}$, where the fiber N^n is a closed compact Spin n-manifold with metric g_N, and the base X^{11-n} is a compact Spin manifold with boundary ∂X^{11-n} and metric g_X. This makes the total space Y^{11} into a manifold with boundary on which we take a family of submersion metrics g_ϵ of the form $g_\epsilon = g_N + \frac{1}{\epsilon^2}\pi^* g_X$ as $\epsilon \to 0$. The Spin bundle of the total space is given in terms of the Spin bundles of the base and the fiber as $S(Y^{11}) = \pi^* S(X^{11-n}) \otimes S(N^n)$. We have a total Dirac operator D_ϵ^Y defined on Y^{11}, a boundary Dirac operator $D_\epsilon^{\partial Y}$ on ∂Y^{11}, and a family of Dirac operators D^N along the fibers. Assuming that the Dirac operators along the fibers N^n are invertible gives that the Dirac operator along the boundary ∂Y^{11} is invertible for small ϵ. This makes the APS problem for the Dirac operator on Y^{11} self-adjoint and so there is an eta-invariant. Let $\eta(D_\epsilon^Y)$ denote the eta-invariant of D_ϵ^Y with the APS boundary conditions. Then the adiabatic limit in this case $\lim_{\epsilon\to 0} \overline{\eta}(D_\epsilon^Y) = \lim_{\epsilon \to 0} \frac{1}{2}\eta(D_\epsilon)$ exists as a real number and, from [12], is given by

$$\lim_{\epsilon\to 0} \overline{\eta}(D_\epsilon) = \int_{X^{11-n}} \widehat{A}(R^X) \wedge \widehat{\eta}\,, \tag{3.7}$$

where R^X is the curvature of g_X and $\widehat{\eta}$ is the (normalized) Bismut-Cheeger η-form. The formula is exactly the same as the case with no boundary [5] – applied to M-theory in [31] and [35] – so that the boundary does not contribute in this case. The above limit can also be taken in the presence of an E_8 vector bundle. In this case we have an analogous conclusion to that of [14]:

Proposition 11 *The dimensional reduction of the phase of the partition function of M-theory on a manifold with boundary is not changed when the base manifold has a boundary. That is , the phase of the partition function in this case is not different from the case when $\partial Y^{11} = \emptyset$.*

II. The fiber with a boundary. Now consider a bundle $N^n \to Y^{11} \xrightarrow{\pi} X^{11-n}$ where the fiber N^n is a compact manifold with boundary ∂N^n. Assume that the vertical tangent bundle $TN^n \subset TY^n$ is equipped with a Spin structure and a Riemannian metric, such that for each fiber N^n the induced metric g^N splits isometrically as $dz^2 + g^{\partial N}$ on $[0,1] \times \partial N^n \subset N^n$. Now consider an E_8 vector bundle $E \to Y^{11}$ with connection ∇^E such that the restriction to $[0,1] \times \partial Y^{11}$ is the pullback of some connection on $E_{\partial Y}$ via the natural projection. Two cases are considered depending on the parity of the dimension of the fiber.

1. Fiber is odd-dimensional. Consider Y^{11} to be the total space of the bundle $N^{2l+1} \to Y^{11} \xrightarrow{\pi} X^{10-2l}$ with odd-dimensional fiber. Y^{11} equipped with an exact b-metric ${}^b g_Y$ (see [26] for how such metrics are defined). Let $x \in C^\infty(M^{10})$ be a distinguished defining function for the boundary $M^{10} = \partial Y^{11}$ so that the metric is of the form ${}^b g_Y = \left(\frac{dx}{x}\right)^2 + g_M$. The corresponding contangent bundle ${}^b T^* Y^{11}$ decomposes orthogonally over the boundary ${}^b T^*_{\partial Y} Y^{11} = \mathbb{R}\left(\frac{dx}{x}\right) \oplus T^* \partial Y^{11}$. Consider a family $\{D_x\}_{x\in X}$ of Dirac operators on the fibers N_x, and let $\{D_0\}$ be the boundary family. From the work of Atiyah-Singer this defines an index class in the K-theory of the base $\mathrm{Ind}(D) \in K^1(X)$. A formula for the corresponding Chern character $\mathrm{ch}(\mathrm{Ind}(D)) \in H^{\mathrm{odd}}(X)$ was conjectured in [6] under the assumption that all the operators induced on the

fiber boundary ∂N are invertible, and under the APS boundary conditions. This is proved in [27] with the invertibility assumption relaxed. Let D be the family of Dirac operators on the fiber and let D_0 be the boundary family (that is on ∂N^{2l+1}). The index class $\mathrm{Index}(D) \in K^1(X)$ has Chern character [27]

$$\mathrm{ch}(\mathrm{Ind}(D, \mathsf{P})) = \int_{Y/X} \widehat{A}(Y/X)\mathrm{ch}(E) - \frac{1}{2}\widehat{\eta}_{\mathrm{odd},\mathsf{P}} \ \in H^{\mathrm{odd}}(X) \tag{3.8}$$

where $\mathrm{ch}(E)$ is the Chern character of the twisting curvature of the vector bundle and P is the spectral section projection.

2. Fiber is even-dimensional. Consider the bundle $N^{2l} \to Y^{11} \xrightarrow{\pi} X^{11-2l}$, where the fiber is even dimensional with a boundary, ∂N^{2l}. As above we assume that the tangent bundle to the fibers $TN^{2l} \subset TY^{11}$ is equipped with a Spin structure. We take the Riemannian metric g^N to be a product $g^N = dz^2 + g^{\partial N}$ on $[0,1] \times \partial N^{2l} \subset N^{2l}$. Consider the Spin bundle along the fiber $S_N = S_N^+ \oplus S_N^-$, where $S_N^{\pm}$ are positive and negative chirality parts. Again take $E \to Y^{11}$ to be an E_8 bundle over Y^{11} with a connection which restricts on $[0,1] \times \partial Y^{11}$ to the pullback of some connection on $E_{\partial Y^{11}}$ via the natural projection (see section 3.1). Consider the twisted Dirac operator along the fiber $D_N^+ : S_N^+ \otimes E \longrightarrow S_N^- \otimes E$ with the Atiyah-Patodi-Singer boundary conditions. Then, from [5] [6], the family D_N^+ determines a continuous family of Fredholm operators with index $\mathrm{Ind}D_N^+$ an element in the K-theory of the base $K^0(X^{11-2l})$ with Chern character

$$\mathrm{ch}(\mathrm{Ind}(D_N^+)) = \int_{N^{2l}} \widehat{A}(R^N)\mathrm{ch}(E) - \widehat{\eta}\,, \tag{3.9}$$

where R^N is the curvature of the connection on TN^{2l}.

Proposition 12 *Consider the partition function for M-theory on a manifold with boundary which of the form of a bundle with base space a closed manifold and a fiber and odd-, respectively, even-dimensional manifold with boundary. Then the phase of the partition function can be reduced to ten dimension via the adiabatic limit to the (logarithm of the) formulas* (3.8) *and* (3.9)*, respectively, provided we impose the appropriate boundary conditions as above.*

3.3 The new Chern-Simons term from the geometric correction to the index

We have discussed the importance of the boundary conditions for fields, bundles and for applying the index theorem for Dirac operators on an eleven-dimensional manifold with boundary. In this section we consider the effect of considering boundary conditions which are not of APS type, that is the manifold is not of the form of a cylinder near the boundary. In a previous paper we have shown that the topological action, and hence the phase of the partition function, can be recast in terms of the signature operator in place of the Dirac operator [38]. Writing the action in terms of the signature allows us to make use of the results of Gilkey [20] to find geometric corrections to the index formula. In this case, the APS index formula can be written as

$$\mathrm{Index}(D, Y^{11}) = \mathrm{Index}(D, Z^{12}) + \overline{\eta}(Y^{11}) + S[Y^{11}]\,, \tag{3.10}$$

where the first and second terms on the right hand side are the usual terms in the APS index formula, that is the index for the case when there is no boundary and the correction from the eta-invariant and zero modes. The third term is an integral over Y^{11} of some Chern-Simons term constructed in [20] and used for the signature operator in [38] for the case when Z^{12} is a disk bundle. This term arises when the metric near the boundary is not a product metric and can be calculated from the connection determined by a suitable choice of the normal to the boundary. The connection ω_Y decomposes into tangential and normal components ω_Y^T and ω_Y^N, respectively, the latter being the second fundamental form. This is covariant under transformations of frames. The difference between characteristic polynomials $P(\Omega_Y)$ and $P(\Omega_Y^T)$ corresponding to the curvatures Ω_Y and Ω_Y^T of the connections ω_Y and ω_Y^T, respectively, is an exact form

$dQ(\Omega_Y, \Omega_Y^T) = P(\Omega_Y) - P(\Omega_Y^T)$, where $Q(\Omega_Y, \Omega_Y^T)$ is a Chern-Simons form. The surface term is then [20] $S[Y^{11}] = -\int_{Y^{11}} Q((\Omega_Y, \Omega_Y^T))$. Explicitly, the eleven-dimensional gravitational Chern-Simons term is

$$L_{CS} = k \int_{Y^{11}} CS_{11}(\omega_Y) = 6k \int_{Y^{11}} \int_0^1 dt \mathrm{Tr} \left[\omega_Y \wedge (td\omega_Y + t^2\omega_Y \wedge \omega_Y) \wedge \cdots \wedge (td\omega_Y + t^2\omega_Y \wedge \omega_Y)\right] \ , \tag{3.11}$$

where k is an integer, a condition from the requirement of independence of the twelve-dimensional extension. Note that the main ingredient in Horava's proposal [21] for a holographic field theory describing M-theory is such a gravitational Chern-Simons action. Thus, we see that this action is part of the action for M-theory that we get from the C-field. We emphasize that we get this gravitational Chern-Simons term in addition to the other terms that we already have in the (exponentiated) action had we not considered the geometric correction to the index formula. We therefore have

Proposition 13 (*i*) *The phase of the M-theory partition function in general boundary conditions leads to gravitational Chern-Simons eleven-form CS_{11} as correction.*
(*ii*) *The holographic Chern-Simons description of M-theory is a phase in the index-theoretic approach.*

Remarks on the case when Y^{11} has a boundary. In contrast to 2+1 dimensions, the higher dimensional Chern-Simons theory does have local, physical degrees of freedom [3]. Again, in the presence of a boundary we take the Chern-Simons term in eleven dimensions as a starting point and ask to which quantity it reduces on the boundary. As pointed out in [21] these correspond to edge states, given by the E_8 super Yang-Mills, on the Horava-Witten boundary and are analogous to similar states in Chern-Simons gauge theory (see for instance [2] and references therein). We make a few remarks for future investigation.

1. It would be interesting to see if the Chern-Simons term we found comes with a fermionic term which makes it supersymmetric. We expect this to be the case as eleven-dimensional supergravity can be written in terms of supersymmetric Chern-Simons theory (see [43]).

2. The geometric correction to the index leading to the Chern-Simons term comes from the second fundamental form [20]. We expect this to be explain the surface term proposed in [30] to modify the Horava-Witten set-up [22]. For the spinors, the obvious boundary condition used in [22] has to be modified to a more elaborate boundary condition, involving projection operators which depend on the gaugino expectation value [30].

3. The theory to which the Chern-Simons theory restricts on the boundary seems to be a higher-dimensional generalization of the WZW model (see [3] [24]). We plan to investigate this elsewhere, building on [36].

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